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# NOTE

## On Characteristic Polynomials of Subspace Arrangements[1]

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ring $\mathbf{k}$ is a finite collection of affine subspaces of $V$. The intersection lattice $L(\mathscr{A})$ is the set of all non-empty intersections of subspaces of $\mathscr{A}$, including $V = \bigcap_{X \in \varnothing} X$ and ordered by inclusion. The characteristic polynomial of $\mathscr{A}$ is defined by

$$\chi(\mathscr{A}, q) = \sum_{X \in L(\mathscr{A})} \mu(X, V)\, q^{\dim X}, \tag{1}$$

where $\mu$ is the Möbius function of $L(\mathscr{A})$. The purpose of this note is to give an interpretation for the name of characteristic polynomial and its values at $\pm 1$. The results are due to Zaslavsky [12], but the method here is quite different.

Given a subset $S$ of $V$, its indicator function $I_S$ is defined by $I_S(x) = 1$ for $x \in S$ and $I_S(x) = 0$ otherwise. For each $X \in L(\mathscr{A})$, define

$$X^0 = X - \bigcup_{Y < X,\ Y \in L(\mathscr{A})} Y.$$

Then $\{X^0 : X \in L(\mathscr{A})\}$ is a collection of disjoint subsets, so that

$$I_X = \sum_{Y \leqslant X,\ Y \in L(\mathscr{A})} I_{Y^0}.$$

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By Möbius inversion we have

$$I_{X^0} = \sum_{Y \leqslant X,\ Y \in L(\mathscr{A})} \mu(Y, X)\, I_Y.$$

In particular, $V^0 = V - \bigcup_{X \in L(\mathscr{A})} X$ and

$$I_{V^0} = \sum_{Y \in L(\mathscr{A})} \mu(Y, V)\, I_Y. \tag{2}$$

This can be viewed as a prototype for the name "characteristic polynomial," but seems not explicitly noticed yet in the study of subspace arrangements. The interpretations made in [1–3] and [6] for values of the characteristic polynomials can be derived from (2) as follows.

Let $L(V)$ denote the lattice of all affine subspaces of $V$ and let $B(V)$ be the Boolean algebra generated by $L(V)$. A *valuation* on $L(V)$ is a set function $v$ from $L(V)$ to an abelian group such that $v(\varnothing) = 0$ and

$$v(X \cup Y) + v(X \cap Y) = v(X) + v(Y)$$

for $X, Y, X \cup Y \in L(V)$. If $v$ is also a valuation on $B(V)$, one can define an integral

$$v(f) = \int f(x)\, dv(x) = \sum_i a_i v(A_i)$$

for any simple function $f = \sum_i a_i I_{A_i}$, $A_i \in B(V)$. Due to Groemer [7], a valuation $v$ on $L(V)$ can be extended to a valuation on $B(V)$ if and only if the following inclusion-exclusion formula

$$v(X_1 \cup \cdots \cup X_n) = \sum_i v(X_i) - \sum_{i,j} v(X_i \cap X_j) + \cdots \tag{3}$$

is satisfied when $X_1 \cup \cdots \cup X_n \in L(V)$.

Now we assume that $V$ is a finite dimensional vector space over a field or division ring $\mathbf{k}$ and we shall give a few interpretations for characteristic polynomials of subspace arrangements.

## 1

Let $\mathbf{k}$ be a finite field $\mathbf{F}_q$ of $q$ elements. The counting measure on $V_{\mathbf{F}_q}$ is obviously a valuation. Applying the counting measure to both sides of (2), we immediately have

$$\chi(\mathscr{A}, q) = \#\left(V_{\mathbf{F}_q} - \bigcup_{X \in \mathscr{A}} X\right). \tag{4}$$

This fact is also observed by Athanasiadis [1] and Björner and Ekedahl [2] (in the special case of hyperplane arrangements it is essentially due to Crapo and Rota [5] and was also observed by Terao [11]).

Let $\mathscr{A}$ be a subspace arrangement whose linear equations have integral coefficients. Then $\mathscr{A}$ induces subspace arrangements $\mathscr{A}_{\mathbf{k}}$ over $\mathbf{k}=\mathbf{F}_q$ ($q$ is a large prime), $\mathbf{R}$ (the field of real numbers), $\mathbf{C}$ (the field of complex numbers), and $\mathbf{H}$ (the division ring of quaternions). The characteristic polynomials of $\mathscr{A}_{\mathbf{F}_q}$, $\mathscr{A}_{\mathbf{R}}$, $\mathscr{A}_{\mathbf{C}}$, and $\mathscr{A}_{\mathbf{H}}$ are related by

$$\chi(\mathscr{A}_{\mathbf{R}}, q)=\chi(\mathscr{A}_{\mathbf{C}}, q^{1/2})=\chi(\mathscr{A}_{\mathbf{H}}, q^{1/4})=\#(V^0_{\mathbf{F}_q}) \tag{5}$$

since, in the defining equation (1), dimension is taken with respect to each specific field $\mathbf{k}$ and not $\mathbf{R}$. The equations in (5) are used to compute the characteristic polynomial in several cases of interest in [1] and relate enumeration in the finite field case to cohomology in the complex case in [2].

Let $V=\mathbf{Z}^n$. Instead of counting points over finite fields, Blass and Sagan [3] interpreted $\mathscr{B}_n$ type arrangements $\mathscr{A}$ by counting lattice points;

$$\chi(\mathscr{A}, q)=\#\left([-s, s]^n\cap\mathbf{Z}^n-\bigcup_{X\in\mathscr{A}} X\cap\mathbf{Z}^n\right), \tag{6}$$

where $s$ is a positive integer and $q=2s+1$. Formula (6) is also a direct consequence of (2) by applying the counting measure to both sides of (2) and is useful in calculating the characteristic polynomials of hyperplane arrangements $\mathscr{A}_n$, $\mathscr{B}_n$, and $\mathscr{D}_n$.

## 2

Let $\mathbf{k}$ be an infinite field. Let us consider the set function $\tau: L(V)\to\mathbf{Z}[q]$ by $\tau(X)=q^{\dim X}$. It is easy to check that (3) is satisfied by induction on $n$. This is simply because $X_1\cup\cdots\cup X_n\in L(V)$ implies that $X_1\cup\cdots\cup X_n=X_i$ for some $i$. Applying $\tau$ to both sides of (2), we have seen that the characteristic polynomial (1) of a subspace arrangement $\mathscr{A}$ is just the extension of the valuation $\tau$ to the complement of $\bigcup\mathscr{A}=\bigcup_{X\in\mathscr{A}} X$. This fact, due to Ehrenborg and Readdy [6], together with (4), show that the characteristic polynomial (1) comes naturally from the characteristic function of the complement of $\bigcup\mathscr{A}$ for a subspace arrangement $\mathscr{A}$. This also explains coincidently the nomenclature of characteristic polynomial.

## 3

Let $\mathbf{k}=\mathbf{R}$. For any relatively open convex polyhedron $P$, define a set function $\chi$ by $\chi(P)=(-1)^{\dim P}$. It is well known that $\chi$ can be uniquely extended to a valuation, called (combinatorial) *Euler characteristic*, on the Boolean algebra generated by all half-spaces $H_{\phi, a}=\{x\in V: \phi(x)>a\}$, where $\phi$ is a linear functional on $V$ and $a$ is a real constant, see, for example, [4, 8]. Let $\mathscr{A}$ be a hyperplane arrangement and let $r(\mathscr{A})$ denote the number of regions in the complement of $\bigcup_{X\in\mathscr{A}} X$. Applying the Euler characteristic $\chi$ to both sides of (2), each cell on the left side contributes $(-1)^{\dim V}$. We immediately have Zaslavsky's formula [12]

$$r(\mathscr{A})=(-1)^{\dim V}\chi(\mathscr{A}, -1).$$

This fact is also observed by Ehrenborg and Readdy [6].

However, there is a *second Euler characteristic* $\bar{\chi}$, due to Schanuel [10], defined by

$$\bar{\chi}(P)=\lim_{r\to\infty}\chi(P\cap B(r))$$

for any polyhedron $P$, where $B(r)$ is the closed ball of radius $r$ centered at the origin. It is immediate that if $P$ is bounded, $\bar{\chi}(P)$ is the same as $\chi(P)$. If $P$ is unbounded, we need Minkowski sum to express $\bar{\chi}(P)$. The *Minkowski sum* of two non-empty sets $A$ and $B$ of $V$ is the set $A+B=\{x+y: x\in A, y\in B\}$. If every element of $A+B$ is uniquely decomposed, we write $A\oplus B$. It is easy to see that each relatively open convex polyhedron $Q$ can be written (not uniquely) as

$$Q=\begin{cases} D\oplus W & \text{if } Q \text{ contains no half-line without containing that whole line}\\ M\oplus W\oplus L & \text{otherwise,}\end{cases} \tag{7}$$

where $D$ is a bounded relatively open convex polyhedron, $W$ is a maximal affine subspace contained in $Q$, $M$ is a polyhedral manifold without boundary and $L$ is an open half-line. Thus $\bar{\chi}(D\oplus W)=(-1)^{\dim D}$ and $\bar{\chi}(M\oplus W\oplus L)=0$. In particular, $\bar{\chi}(X)=1$ for all affine subspaces $X$. For a detailed study of $\bar{\chi}$, see [4].

Now if not all hyperplanes in $\mathscr{A}$ are parallel to a line, then every unbounded cell in the complement of $\bigcup\mathscr{A}$ contains a half-line, but not that whole line, so its second Euler characteristic is zero. Applying $\bar{\chi}$ to both sides of (2), we obtain again Zaslavsky's formula [12] on the number of bounded regions in the complement of $\bigcup\mathscr{A}$:

$$b(\mathscr{A})=(-1)^{\dim V}\chi(\mathscr{A}, 1). \tag{8}$$

Notice that if the aforementioned condition is not satisfied, that is, all hyperplanes are parallel to a line $\ell$ through the origin, then every region has $\ell$ as a direct summand; so every region is unbounded; thus (8) is obviously incorrect. In this case, however, $|\chi(\mathscr{A}, 1)|$ counts the number of regions of type $D \oplus W$ in (7) and all regions in the complement have $W$ as a direct summand. Therefore $W$ is the recession space of $\mathscr{A}$. In general we define the *recession space* rec $\mathscr{A}$ to be the maximal vector subspace that is parallel to all hyperplanes of $\mathscr{A}$. Let $H$ be an affine subspace such that $V = H \oplus \operatorname{rec} \mathscr{A}$. Define the hyperplane arrangement $\mathscr{A}_H = \{X \cap H : X \in \mathscr{A}\}$ of $H$. Let $b(\mathscr{A}_H)$ denote the number of bounded regions in $H - \bigcup_{X \in \mathscr{A}_H} X$. Then we have

$$b(\mathscr{A}_H) = (-1)^{\dim H} \chi(\mathscr{A}_H, 1) = (-1)^{\dim V - \dim \operatorname{rec} \mathscr{A}} \chi(\mathscr{A}, 1). \tag{9}$$

It is these regions which are often referred to as “bounded” or “relatively bounded” in the literature, without much more explanation. We caution the reader about the sign in (9) since it is often carelessly misquoted.

Let $f_k(\mathscr{A})$ denote the number of $k$-cells in $V$ cut by $\mathscr{A}$ and let $b_k(\mathscr{A}_H)$ denote the number bounded cells in $H$ cut by $\mathscr{A}_H$. Then

$$f_k(\mathscr{A}) = \sum_{\substack{X \in L(\mathscr{A}) \\ \dim X = k}} (-1)^k \chi(\mathscr{A}_X, -1);$$

$$b_k(\mathscr{A}_H) = \sum_{\substack{X \in L(\mathscr{A}) \\ \dim(X \cap H) = k}} (-1)^k \chi(\mathscr{A}_{X \cap H}, 1).$$

## 4

Let $\mathbf{k} = \mathbf{C}$ or $\mathbf{H}$. The *Poincaré* polynomial is defined by

$$P(\mathscr{A}, q) = \sum_{k \geqslant 0} q^k \dim H^k(V - \textstyle\bigcup \mathscr{A}, \mathbf{R}).$$

It is known that $P(\mathscr{A}, q) = q^{\dim V} \chi(\mathscr{A}, -q^{-1})$, see [9]. Thus the Euler characteristics of the complements of $\bigcup \mathscr{A}_{\mathbf{C}}$ and $\bigcup \mathscr{A}_{\mathbf{H}}$ are given by

$$\chi\left(V_{\mathbf{C}} - \bigcup_{X \in \mathscr{A}_{\mathbf{C}}} X\right) = \chi\left(V_{\mathbf{H}} - \bigcup_{X \in \mathscr{A}_{\mathbf{H}}} X\right) = \chi(\mathscr{A}, 1). \tag{10}$$

This last formula (10) can be directly read from (1) because each element of $L(\mathscr{A})$ is an even dimensional affine subspace. If rec $\mathscr{A}_{\mathbf{R}}$ is the trivial $\{o\}$, then they are equal to the number of bounded cells in the complement of $\bigcup \mathscr{A}_{\mathbf{R}}$.

## REFERENCES


1. C. A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, *Adv. in Math.* **122** (1996), 193–233.
2. A. Björner and T. Ekedahl, Subspace arrangements over finite fields: Cohomological and enumerative aspects, *Adv. in Math.* **129** (1997), 159–187.
3. A. Blass and B. Sagan, Characteristic and Ehrhart polynomials, *J. Algebraic Combin.* **7** (1998), 115–126.
4. B. Chen, On the Euler characteristics of finite unions of convex sets, *Discrete Comput. Geom.* **10** (1993), 79–93.
5. H. Crapo and G.-C. Rota, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," M.I.T. Press, Cambridge, MA, 1970.
6. R. Ehrenborg and M. Readdy, On valuations, the characteristic polynomial, and complex subspace arrangements, *Adv. in Math.* **134** (1998), 32–42.
7. H. Groemer, On the extension of additive functionals on the classes of convex sets, *Pacific J. Math.* **75** (1978), 397–410.
8. D. A. Klain and G.-C. Rota, "Introduction to Geometric Probability," Cambridge Univ. Press, Cambridge, UK, 1997.
9. P. Orlik and H. Terao, "Arrangements of Hyperplanes," Grundlehren, Vol. 300, Springer-Verlag, New York, 1992.
10. S. Schanuel, Negative sets have Euler characteristic and dimension, *in* "Lecture Notes in Mathematics," Vol. 1488, Springer-Verlag, New York, 1991.
11. H. Terao, The Jacobians and the discriminants of finite reflection groups, *Tôhoku Math. J.* **41** (1989), 237–247.
12. T. Zaslavsky, Facing up to arrangements: Face-counting formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.* **1**, No. 154 (1975).